TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 217, 1976

ADJOINT ABELIAN OPERATORS ON L^p AND $C(K)$

BY

RICHARD J. FLEMING AND JAMES E. JAMISON

ABSTRACT. An operator A on a Banach space X is said to be adjoint abelian if there is a semi-inner product $[\cdot, \cdot]$ consistent with the norm on X such that $[Ax, y] = [x, Ay]$ for all $x, y \in X$. In this paper we show that every adjoint abelian operator on $C(K)$ or $L^p(\Omega, \Sigma, \mu)$ $(1 < p < \infty, p \neq 2)$ is a multiple of an isometry whose square is the identity and hence is of the form $Ax(\cdot) = \lambda\alpha(\cdot)(x \circ \phi)(\cdot)$ where α is a scalar valued function with $\alpha(\cdot)\alpha \circ \phi(\cdot) = 1$ and ϕ is a homeomorphism of K (or a set isomorphism in case of $L^p(\Omega, \Sigma, \mu)$) with $\phi \circ \phi =$ identity (essentially).

1. **Introduction.** An operator A on a Banach space X is said to be *adjoint abelian* if there is a semi-inner product $[\cdot, \cdot]$ consistent with the norm on X such that

$$[Ax, y] = [x, Ay] \tag{1}$$

for all $x, y \in X$. In this note we show that every adjoint abelian operator on $C(K)$ or L^p $(1 < p < \infty, p \neq 2)$ is a multiple of an isometry and hence is of the form

$$Ax(\cdot) = \lambda\alpha(\cdot)(x \circ \phi)(\cdot) \tag{2}$$

where α is a scalar valued function with $\alpha(\cdot)\alpha \circ \phi(\cdot) = 1$ and ϕ is a homeomorphism of K (or a set isomorphism in the case of L^p) with $\phi \circ \phi =$ identity (essentially).

Our method is to use known characterizations of Hermitian operators on the spaces in question, together with the observation of Stampfli [12] that if A is adjoint abelian then A^2 is both Hermitian and adjoint abelian, to show that if A is adjoint abelian then (Theorems 1 and 4)

$$A^2 = \rho I \quad \text{for some } \rho > 0. \tag{3}$$

Furthermore, if (3) holds, then for $\lambda = \sqrt{\rho}$ we have

$$\begin{aligned}\|(\lambda^{-1}A)(x)\|^2 &= [\lambda^{-1}Ax, \lambda^{-1}Ax] = (\lambda^{-1})^2[Ax, Ax] \\ &= \rho^{-1}[A^2x, x] = \rho^{-1}[\rho x, x] = [x, x] = \|x\|^2.\end{aligned}$$

Presented to the Society, January 15, 1974; received by the editors November 19, 1973.
AMS (MOS) subject classifications (1970). Primary 47B99, 46E30.
Key words and phrases. Adjoint abelian, isometries, semi-inner products, reflections, scalar operators.

Copyright © 1976, American Mathematical Society

Hence $\lambda^{-1}A$ is an isometry. The result (2) now follows easily from known characterizations of isometries on $C(K)$ and L^p.

The result of the calculation above will be used in both §§2 and 3 and we find it convenient to include a formal statement here.

LEMMA 1. *If A is adjoint abelian on a Banach space X and $A^2 = \rho I$ for some $\rho > 0$, then A is a positive multiple of an isometry on X.*

In [12], Stampfli asked whether every adjoint abelian operator on a weakly complete Banach space is scalar. In [11], it was shown that if A satisfies (3), then A is a scalar operator. Hence by our Theorems 1 and 4, every adjoint abelian operator on $C(K)$ (K compact metric) or $L^p(\Omega, \Sigma, \mu)$ ((Ω, Σ, μ) a σ-finite measure space) is a scalar operator. (A slightly more general such result is given in §4.)

2. **Adjoint abelian operators on $C(K)$.** Let K be a compact metric space and let $C(K)$ denote the space of continuous complex valued functions on K with the supremum norm. Any semi-inner product $[\cdot, \cdot]$ on $C(K)$ is determined by a mapping $f \longrightarrow f^*$ of $C(K)$ to its dual, where $f^*(f) = \|f\|^2$, $\|f^*\| = \|f\|$ and $[g, f] = f^*(g)$. For each such $f^* \in C(K)^*$, there exists a regular complex Borel measure ν_f on the Borel sets of K such that $f^*(g) = \int_K g\, d\nu_f$ [3]. Hence

$$[g, f] = \int_K g\, d\nu_f \tag{4}$$

is the general form of any s.i.p. in $C(K)$. We note here that the measure ν_f really depends on f^* so that in the notation of (4) we are assuming that the particular mapping $f \longrightarrow f^*$ has been preassigned.

For $f \in C(K)$, let $P_f = \{t \in K: |f(t)| = \|f\|\}$ be called the *peak set* for f.

LEMMA 2. *For $f \in C(K)$, the measure ν_f as in* (4) *has the properties that*
(i) $|\nu_f|(P_f) = \|f\|$,
(ii) *if $f \geq 0$, then $\nu_f(P_f) = \|f\|$,*
(iii) *if $P_f = \{t_0\}$, then $\nu_f(P_f) = \bar{f}(t_0)$.*

PROOF. Suppose $f \in C(K)$ and let G be any open set in K containing P_f. Let $\lambda = \sup\{|f(x)|: x \in K\backslash G\}$. Now $\lambda < \|f\|$ and if $|\nu_f|(K\backslash G) > 0$, then

$$\|f\|^2 = [f, f] = \left|\int_K f\, d\nu_f\right| \leq \lambda|\nu_f|(K\backslash G) + \|f\|\ |\nu_f|(G)$$
$$< \|f\|(|\nu_f|(K\backslash G) + |\nu_f|(G)) = \|f\|^2$$

since $|\nu_f|(K) = \|f^*\| = \|f\|$. Hence we must have $|\nu_f|(K\backslash G) = 0$ and $|\nu_f|(G) = \|f\|$.

Since ν_f is regular, for $\epsilon > 0$, there exists an open set G such that $P_f \subset G$

and $|\nu_f|(G) < |\nu_f|(P_f) + \epsilon$. We conclude that $|\nu_f|(P_f) = \|f\|$. If B is any Borel set such that $B \cap P_f = \emptyset$, there is a closed set $F \subset B$ such that $|\nu_f|(B) < |\nu_f|(F) + \epsilon$. The argument above shows that $|\nu_f|(K\backslash F) = 1$ so that $|\nu_f|(F) = 0$. It follows that $|\nu_f|(B) = 0$. Hence, for $f \geqslant 0$, we have

$$\|f\|^2 = \int_K f d\nu_f = \int_{P_f} f\, d\nu_f + \int_{K\backslash P_f} f\, d\nu_f = \int_{P_f} f\, d\nu_f = \|f\|\nu_f(P_f).$$

Finally, suppose $P_f = \{t_0\}$. Then $\|f\|^2 = \int_K f\, d\nu_f = f(t_0)\nu_f(\{t_0\})$ so that $|f(t_0)|^2 = f(t_0)\nu_f(t_0)$. Therefore $\nu_f(\{t_0\}) = \overline{f}(t_0)$ and the proof is complete.

We observe here that if ψ is a function which assigns to each $g \in C(K)$ an element of the peak set P_g, then

$$[f, g] = f(\psi(g))\overline{g}(\psi(g)) \tag{5}$$

defines a s.i.p. on $C(K)$ which is compatible with the norm. If ϕ is a homeomorphism of K onto itself with the property that $\phi \circ \phi$ is the identity on K, then $\|g \circ \phi\| = \|g\|$ and $P_{g\circ\phi} = \phi(P_g)$ for all $g \in C(K)$.

LEMMA 3. *If ϕ is a homeomorphism of the compact metric space K with the property that $\phi \circ \phi$ is the identity, then there is a choice function ψ_0 as in* (5) *such that*

$$\psi_0(g \circ \phi) = \phi(\psi_0(g)) \tag{6}$$

and

$$\psi_0(g_1) = \psi_0(g_2) \quad \text{whenever } P_{g_1} = P_{g_2} \tag{7}$$

for all $g, g_1, g_2 \in C(K)$.

PROOF. Let $\mathcal{C}$ be the set of all choice functions on subsets $C(K)$ with the following properties:

(i) If $\psi \in \mathcal{C}$, then $\mathcal{D}(\psi)$ = domain of ψ is of the form $Y \cup (Y \circ \phi)$ for some subset $Y \subset C(K)$;

(ii) $\psi(g \circ \phi) = \phi(\psi(g))$ for each $g \in \mathcal{D}(\psi)$;

(iii) $\psi(g_1) = \psi(g_2)$ whenever $g_1, g_2 \in \mathcal{D}(\psi)$ and $P_{g_1} = P_{g_2}$.

If $\psi_1, \psi_2 \in \mathcal{C}$ define $\psi_1 \leqslant \psi_2$ whenever $\mathcal{D}(\psi_1) \subset \mathcal{D}(\psi_2)$ and $\psi_2(f) = \psi_1(f)$ for $f \in \mathcal{D}(\psi_1)$. Then $\mathcal{C}$ is a nonempty partially ordered set and it may be shown by an argument using Zorn's Lemma that $\mathcal{C}$ contains a maximal element ψ_0. It is straightforward to show that $\mathcal{D}(\psi_0) = C(K)$ and therefore (ii) and (iii) are the same as (6) and (7).

As we mentioned in the introduction, we will need a characterization of Hermitian operators on $C(K)$. Sinclair [10] has shown that an operator is Hermitian on $C(K)$ if and only if it is multiplication by a real valued function in $C(K)$. This result has also been obtained in [14] by a different method.

THEOREM 1. *Let K be a compact metric space and suppose $A \neq 0$ is an adjoint abelian operator on $C(K)$. Then there exists a positive constant λ such that $A^2 = \lambda I$ (where I is the identity operator).*

PROOF. Since A is adjoint abelian, there exists for each $f \in C(K)$ a regular complex Borel measure ν_f such that $[g, f] = \int_K g\, d\nu_f$ defines a s.i.p. on $C(K)$ compatible with the norm and such that

$$[Ag, f] = [g, Af] \quad \text{for all } g, f \in C(K). \tag{8}$$

Now A^2 must also satisfy (8) and must be Hermitian as well, i.e. $[A^2 f, f]$ is real for all $f \in C(K)$ [12]. Thus by the characterization of Hermitian operators mentioned above, there exists a real valued function $h \in C(K)$ such that $A^2 f = hf$ for all $f \in C(K)$. In fact, $h(t) \geqslant 0$ for all t.

Let $t_0 \in K$. Suppose $|h(t_0)| < \|h\|$ and let $t_h \in P_h$. By Urysohn's lemma, there exists $g \in C(K)$ such that $g(t_0) = 1$, $g(t_h) = \frac{1}{2}(1 + |h(t_0)|/\|h\|) = \lambda_0$ and $g(t) \in (\lambda_0, 1)$ for all $t \in K\{t_0, t_h\}$. Then

$$\begin{aligned} |hg(t_h)| &= |h(t_h)|\, |g(t_h)| = (\|h\|/2)(1 + |h(t_0)|/\|h\|) \\ &= \tfrac{1}{2}(\|h\| + |h(t_0)|) > |h(t_0)| = |hg(t_0)|. \end{aligned}$$

Hence $t_0 \notin P_{hg}$. Again by Urysohn's lemma there exists $f \in C(K)$ such that $f(t_0) = 1$ and $f(t) = \|h\|/\|hg\|$ for $t \in P_{hg}$. Since g has a singleton peak set, we recall from Lemma 2(iii) that

$$[A^2 f, g] = [hf, g] = \int_K hf\, d\nu_g = hf(t_0)\bar{g}(t_0) = hf(t_0) = h(t_0).$$

Moreover,

$$\begin{aligned} [f, A^2 g] &= [f, hg] = \int_K f\, d\nu_{hg} = \int_{P_{hg}} f\, d\nu_{hg} \\ &= \frac{\|h\|}{\|hg\|} \nu_{hg}(P_{hg}) = \frac{\|h\|}{\|hg\|} \|hg\| = \|h\| \end{aligned}$$

since $\nu_{hg}(P_{hg}) = \|hg\|$ by Lemma 2(ii). Therefore $[A^2 f, g] \neq [f, A^2 g]$ which is a contradiction. We conclude that h is constant; indeed, $A^2 f = \|h\| f$ for all $f \in C(K)$.

THEOREM 2. *Let K be a compact metric space and A a nonzero operator on $C(K)$. Then A is adjoint abelian if and only if there exists a homeomorphism ϕ on K, a positive constant λ and a unimodular function $\alpha \in C(K)$ such that for every $f \in C(K)$,*

$$Af(t) = \lambda\alpha(t)f \circ \phi(t), \qquad t \in K, \tag{9}$$

where

(i) $(\phi \circ \phi)(t) = t$ *for all* $t \in K$, *and*

(ii) $\alpha(t)\alpha(\phi(t)) = 1$ *for all* $t \in K$.

PROOF. Let us first prove the sufficiency of the conditions.

Let a s.i.p. be given as in (5) where the associated choice function ψ satisfies (6) and (7). Then

$$[Af, g] = \lambda\alpha(\psi(g))f \circ \phi(\psi(g))\overline{g}(\psi(g)),$$

and since $P_{Ag} = P_{g \circ \phi}$ we have

$$\psi(Ag) = \psi(g \circ \phi) = \phi(\psi(g)). \tag{10}$$

By using (10) along with (9), (i) and (ii), we may then obtain

$$\begin{aligned}[f, Ag] &= f(\psi(Ag))\overline{Ag}(\psi(Ag)) = \lambda f \circ \phi(\psi(g))\overline{\alpha}(\phi(\psi(g)))\overline{g \circ \phi}(\phi(\psi(g))) \\ &= \lambda(f \circ \phi)(\psi(g))\alpha(\psi(g))\overline{g}(\psi(g)) = [Af, g].\end{aligned}$$

On the other hand, suppose A is adjoint abelian. By Theorem 1, $A^2 = \rho I$ for some $\rho > 0$ and by Lemma 1, $A = \lambda U$ for some isometry U on $C(K)$. By the Banach-Stone Theorem [3, p. 442], there exists a unimodular function α and a homeomorphism ϕ on K such that $Uf(\cdot) = \alpha(\cdot)f \circ \phi(\cdot)$ for all $t \in C(K)$. Thus (9) is satisfied.

Since $A^2 = \lambda^2 I$, we have

$$\begin{aligned}\lambda^2 f(t) &= (A^2 f)(t) = A(\lambda\alpha f \circ \phi)(t) \\ &= \lambda^2\alpha(t)\alpha(\phi(t))f(\phi \circ \phi(t)) \quad \text{for all } t \in K \text{ and } f \in C(K).\end{aligned} \tag{11}$$

If we take $f \equiv 1$ in (11) we obtain

$$1 = \alpha(t)\alpha(\phi(t)) \quad \text{for all } t \in K.$$

From this and (11) we get

$$f(t) = f(\phi \circ \phi(t)) \quad \text{for all } f \in C(K) \text{ and } t \in K. \tag{12}$$

It follows readily that $(\phi \circ \phi)(t) = t$ for all $t \in K$ which establishes (i) and concludes the proof of the theorem.

We remark that in the case of $X = [0, 1]$, for example, there are only two choices for ϕ

$$\phi(t) = t \quad \text{and} \quad \phi(t) = 1 - t.$$

3. **Adjoint abelian operators on L^p.** In this section we characterize the adjoint abelian operators on $L^p(\Omega, \Sigma, \mu)$ where (Ω, Σ, μ) is a σ-finite measure

space. For this we first need a characterization of Hermitian operators on these spaces. In the case that (Ω, Σ, μ) is nonatomic the result is given in [8].

A s.i.p. compatible with the norm in $L^p(\Omega, \Sigma, \mu)$ for $1 \leqslant p < \infty$, $p \neq 2$, is given by

$$[f, g] = \|g\| \int_\Omega f\left(\frac{|g|}{\|g\|}\right)^{p-1} \operatorname{sgn} g. \tag{13}$$

If $p > 1$, this s.i.p. is unique, but for the characterization of Hermitians any s.i.p. compatible with the norm will suffice.

LEMMA 4. *Let f_1, f_2 be real valued functions in $L^p(\Omega, \Sigma, \mu)$, $1 \leqslant p < \infty$, $p \neq 2$, with essentially disjoint supports Ω_1 and Ω_2 respectively. Then*

$$\int_\Omega (Hf_2)|f_1|^{p-1} \operatorname{sgn} f_1 = \overline{\int_\Omega (Hf_1)|f_2|^{p-1} \operatorname{sgn} f_2}$$

for every Hermitian operator H on L^p.

PROOF. The result follows immediately from a result of Tam [13] and the fact that if H is Hermitian, $[H(f_1 + e^{i\theta} f_2), (f_1 + e^{i\theta} f_2)]$ is real for every real value of θ.

COROLLARY 1. *If $\Omega_1, \Omega_2 \in \Sigma$ with $\mu(\Omega_1 \cap \Omega_2) = 0$, and χ_1, χ_2 are the associated characteristic functions, then*

$$\int_{\Omega_1} H\chi_2 = \overline{\int_{\Omega_2} H\chi_1}.$$

THEOREM 3. *Let H be a Hermitian operator on $L^p(\Omega, \Sigma, \mu)$ with $1 \leqslant p < \infty$, $p \neq 2$. Then H is Hermitian if and only if there exists a real valued function $h \in L^\infty(\Omega, \Sigma, \mu)$ such that $Hf = hf$ a.e. for every $f \in L^p$.*

PROOF. The proof will be given for a finite measure space since the extension to σ-finite measure spaces follows exactly as indicated in §6 of Lumer's paper [8].

Let $\Omega_1 \in \Sigma$ and χ_1 be its characteristic function. Suppose $H\chi_1 \neq 0$ a.e. on $\Omega \backslash \Omega_1$. Then there exists a measurable set $\Omega_2 \subset \Omega \backslash \Omega_1$ with positive measure such that

$$\int_{\Omega_2} H\chi_1 \neq 0. \tag{14}$$

Let χ_2 be the characteristic function of Ω_2 and $f_1 = \alpha\chi_1$, $f_2 = \chi_2$ with $\alpha > 1$. Applying Lemma 4 we obtain

$$\int_{\Omega_1} (H\chi_2)\alpha^{p-1} = \alpha \overline{\int_{\Omega_2} H\chi_1}. \tag{15}$$

It now follows from Corollary 1 and (15) that

$$(\alpha^{p-1} - \alpha) \int_{\Omega_2} H\chi_1 = 0 \tag{16}$$

which contradicts (14). Hence, $H\chi_1 = 0$ a.e. on $\Omega \backslash \Omega_1$ and the proof now follows exactly as the proof of Theorem 9 in [8].

THEOREM 4. *Let A be adjoint abelian on $L^p(\Omega, \Sigma, \mu)$ where $1 < p < \infty$, $p \neq 2$. Then there exists a positive constant ρ such that $A^2 = \rho I$.*

PROOF. As we have previously observed, A^2 is Hermitian as well as adjoint abelian. Hence by Theorem 3 there exists a real L^∞ function h such that $A^2 f = hf$ for every $f \in L^p$ and where $h(t) \geq 0$ a.e. on Ω ($[hf, f] = [Af, Af] \geq 0$). Furthermore, A^2 is adjoint abelian; thus

$$(17) \qquad [hf, g] = [f, hg] \quad \text{for all } f, g \in L^p.$$

From the combination of (13) and (17) we obtain

$$(18) \qquad \int_\Omega f \operatorname{sgn} g \left[\|g\| h \left(\frac{|g|}{\|g\|} \right)^{p-1} - \|hg\| \left(\frac{|hg|}{\|hg\|} \right)^{p-1} \operatorname{sgn} h \right] = 0$$

for all $f, g \in L^p$. For a given f, g we may replace f by an appropriate product of the form $e^{i\alpha(t)} f(t)$ so that (18) holds with the integrand replaced by its absolute value. Hence,

$$(19) \qquad |f| \, | \, \|g\| h (|g|/\|g\|)^{p-1} - \|hg\| (|hg|/\|hg\|)^{p-1} \operatorname{sgn} h| = 0 \quad \text{a.e.}$$

Let $Z(k) = \{t \in \Omega: k(t) = 0\}$ for any function k on Ω. It follows from (19) that

$$(20) \qquad |f| \, |g|^{p-1} |1/\|g\|^{p-2} - h^{p-2}/\|hg\|^{p-2}| = 0 \quad \text{a.e.}$$

on $\Omega \backslash Z(h)$. For any $g \in L^p$, we have (taking $f = g$)

$$(21) \qquad |1/\|g\|^{p-2} - h^{p-2}/\|hg\|^{p-2}| = 0 \quad \text{a.e.}$$

on $\Omega \backslash (Z(h) \cup Z(g))$. Therefore

$$(22) \qquad h(t) = \|hg\|/\|g\| \quad \text{a.e.}$$

on $\Omega \backslash (Z(g) \cup Z(h))$ for all $g \in L^p$. It follows that h is constant a.e. in $\Gamma \backslash Z(h)$ for every $\Gamma \in \Sigma$ with $\mu(\Gamma) < \infty$. Since Ω is σ-finite, it follows that h is constant a.e. on $\Omega \backslash Z(h)$. In fact from (22) we must have

$$(23) \qquad h = \|gh\|/\|g\| = \lambda \quad \text{a.e.}$$

on $\Omega \backslash Z(h)$ for every $g \in L^p$. The proof of the theorem will be complete if we can show that $\mu(Z(h)) = 0$.

Let $F \subset Z(h)$ with $F \in \Sigma$; $\mu(F) < \infty$ and $G \subset \Omega \backslash Z(h)$ with $\sigma < \mu(G) < \infty$. Such sets F, G must exist; otherwise $Z(h)$ and $\Omega \backslash Z(h)$ would be atoms of infinite measure which is impossible since the measure space is σ-finite. Let $g = \chi_G + \chi_F$ so that $\|g\|^p = \mu(G) + \mu(F)$. Now

$$gh = h\chi_G = \lambda\chi_G \quad \text{and} \quad \lambda^p = \frac{\|gh\|^p}{\|g\|^p} = \frac{\lambda^p \mu(G)}{\mu(G) + \mu(F)} \quad \text{by (23).}$$

It now follows that $\mu(\overline{Z(h)}) = 0$.

We may use Lamperti's characterization of onto isometries of L^p to obtain the description of adjoint abelian operators on L^p announced in the introduction. Let us recall the notation and the theorem of Lamperti which we shall need [6].

A regular set isomorphism of the measure space (Ω, Σ, μ) will mean a mapping T of Σ into Σ defined modulo sets of measure zero satisfying $T(\Omega \backslash F) = T(\Omega)\backslash TF$, $T(\bigcup F_n) = \bigcup TF_n$ disjoint F_n, and $\mu(TF) = 0$ if and only if $\mu(F) = 0$. For any measurable function f on (Ω, Σ, μ) we will write $f \circ T$ to be the function obtained from a limit of simple functions where by definition, $\chi_E \circ T = \chi_{TE}$ for each $E \in \Sigma$. Lamperti [6] proved that if U is an isometry on L^p, then there exists a regular set isomorphism T and a function $\alpha(t)$ such that

$$Uf(t) = \alpha(t) f \circ T(t) \quad \text{a.e.} \tag{24}$$

and

$$\int_{TE} |\alpha|^p \, d\mu = \mu(E) \quad \text{for } E \in \Sigma. \tag{25}$$

Conversely, if U satisfies (24) and α satisfies (25), then U is an isometry of L^p.

THEOREM 5. *If $1 < p < \infty$, $p \neq 2$, and A is a nonzero operator on L^p, then A is adjoint abelian if and only if there exists a regular set isomorphism T, a measurable function α and a real number λ such that*

$$Af(t) = \lambda\alpha(t) f \circ T(t) \quad \text{a.e. for } f \in L^p \tag{26}$$

where

$$\alpha(t)\alpha \circ T(t) = 1 \quad \text{a.e.,} \tag{27}$$

$$\int_{TE} |\alpha|^p \, d\mu = \mu(E) \quad \text{for } E \in \Sigma, \tag{28}$$

$$T \circ T(E) = E \quad (\text{modulo sets of measure zero}). \tag{29}$$

PROOF. If A is adjoint abelian, then by Theorem 4 and Lemma 1, $A = \lambda U$ where U is an isometry and λ is real. By Lamperti's theorem, there is an α and a regular set isomorphism T so that (24) and (25) are satisfied for U; hence (26) and (28) must hold. Since $A^2 = \lambda^2 I$, we have from (26) that

$$\lambda^2 f(t) = (A^2 f)(t) = \lambda^2 \alpha(t)\alpha \circ T(t) f \circ T \circ T(t) \quad \text{a.e.} \tag{30}$$

for each $f \in L^p$. If E is any subset of finite measure, we may take $f = \chi_E$ so that

$$\chi_E(t) = \alpha(t)\alpha \circ T(t)\chi_{T \circ T(E)} \quad \text{a.e.}$$

It follows from this that $T \circ T(E) = E$ modulo a set of measure zero and $\alpha(t)\alpha \circ T(t) = 1$ a.e. on sets of finite measure. The extension to sets of infinite measure follows readily from the σ-finiteness of Ω, and (27), (29) are established.

Next suppose (26), (27), (28), and (29) are satisfied by A, α, T, λ. If $E \in \Sigma$, we have

$$\begin{aligned}\int_\Omega \chi_E \circ T &= \int_\Omega \chi_{TE} = \mu(TE)\\ &= \int_{T \circ T(E)} |\alpha|^p \quad \text{by (28)}\\ &= \int_E |\alpha|^p \quad \text{by (29)}\\ &= \int_\Omega |\alpha|^p \chi_E.\end{aligned}$$

In the same way, it can be shown that if f is a simple function with support of finite measure then

$$\int_\Omega f \circ T = \int_\Omega |\alpha|^p f$$

and finally, if $|\alpha|^p f$ is integrable, then $f \circ T$ is integrable and

$$\int_\Omega f \circ T = \int_\Omega |\alpha|^p f \quad \text{for all } f \in L^p \tag{31}$$

since any measurable f is the a.e. limit of a sequence of simple functions with finite support [9, p. 224].

Now suppose $f, g \in L^p$. Then using the given conditions along with (31) and the fact that T distributes across products, we obtain

$$\begin{aligned}[Af, g] &= \lambda \|g\| \int \alpha(f \circ T) \left(\frac{|g|}{\|g\|}\right)^{p-1} \operatorname{sgn} g\\
&= \lambda \|g\| \int \alpha f \circ T |\alpha|^{p-1} |\alpha \circ T|^{p-1} \left(\frac{|g|}{\|g\|}\right)^{p-1} \operatorname{sgn} \alpha \operatorname{sgn}(\alpha \circ T) \operatorname{sgn} g\\
&= \lambda \|g\| \int |\alpha|^p (f \circ T) |\alpha \circ T|^{p-1} \left(\frac{|g|}{\|g\|}\right)^{p-1} \operatorname{sgn}(\alpha \circ T) \operatorname{sgn} g\\
&= \|Ag\| \int (f \circ T \circ T) |\alpha \circ T \circ T|^{p-1} \left(\frac{|g \circ T|}{\|g\|}\right)^{p-1}\\
&\qquad \cdot \frac{\lambda}{|\lambda|} \operatorname{sgn}(\alpha \circ T \circ T) \operatorname{sgn}(g \circ T)\\
&= \|Ag\| \int f \left(\frac{|\lambda|^{p-1} |\alpha|^{p-1} |g \circ T|^{p-1}}{|\lambda|^{p-1} \|g\|^{p-1}}\right) \frac{\lambda}{|\lambda|} \operatorname{sgn} \alpha \operatorname{sgn}(g \circ T)\\
&= \|Ag\| \int f \left(\frac{|Ag|}{\|Ag\|}\right)^{p-1} \operatorname{sgn} Ag\\
&= [f, Ag],\end{aligned}$$

and A is adjoint abelian.

A characterization of adjoint abelian operators on l^p which is included in Theorem 5, has been obtained previously in [1] and [4].

The results above are related to some recent work of Byrne and Sullivan [2] on contractive projections on L^p. A projection P on L^p is called *contractive* if $\|P\| = 1$. An isometry U with the property that $U^2 = I$ is called a *reflection.* In [2], it is proved that P and $I - P$ are contractive if and only if $P = (I + U)/2$ for some reflection U. The next two corollaries are immediate from Theorems 4 and 5 and the work in [2].

COROLLARY 2. *A nonzero operator A on L^p is adjoint abelian if and only if A is a real multiple of a reflection.*

COROLLARY 3. *Both P and $I - P$ are contractive projections on L^p if and only if there is a real number λ and an adjoint abelian operator A such that $P = (I + \lambda A)/2$.*

Stampfli [12] has proved that an operator B on a weakly complete Banach space has a proper invariant subspace if it commutes with an adjoint abelian operator A where $A \neq \lambda I$.

COROLLARY 4. *Let B be a bounded operator $L^p(\Omega, \Sigma, \mu)$ where $1 < p < \infty$, $p \neq 2$. If there exists a regular, measure preserving set isomorphism T such that T is not the identity, $T \circ T(E) = E$ modulo sets of measure zero for all $E \in \Sigma$, and*

$$B(f \circ T) = Bf \circ T \quad \text{for all } f \in L^p, \tag{32}$$

then B has a proper invariant subspace.

PROOF. If we define A on L^p by $Af = f \circ T$, then (26), (27), (28), and (29) are satisfied by taking $\alpha \equiv 1$. Hence A is adjoint abelian and (32) is simply the condition that A commutes with B.

4. Adjoint abelian operators and isometries. We have shown that adjoint abelian operators on $C(K)$ and $L^p(\Omega, \Sigma, \mu)$ are multiples of isometries. This is also the case for adjoint abelian operators on certain spaces of class S discussed in [4]. The next theorem characterizes the types of isometries which can give rise to adjoint abelian operators in this manner.

THEOREM 6. *Let U be an isometry on the Banach space X and λ a real scalar. The operator $A = \lambda U$ is adjoint abelian if and only if $U^2 = I$.*

PROOF. Let U be an isometry with $U^2 = I$. By a theorem of Koehler and Rosenthal [5] there exists a s.i.p $[\cdot, \cdot]$ compatible with the norm such that

$$[Ux, Uy] = [x, y] \quad \text{for all } x, y \in X. \tag{33}$$

Hence,

$$[\lambda Ux, y] = [\lambda Ux, U^2 y]$$
$$= [\lambda x, Uy] \quad \text{from (33)}$$
$$= [x, \lambda Uy].$$

Next suppose $A = \lambda U$ is adjoint abelian and U is an isometry. Then for every $x \in X$

(34) $$[(\lambda U)^2 x, x] = [\lambda Ux, \lambda Ux] = \|\lambda Ux\|^2 = \lambda^2 \|x\|^2.$$

It follows from (34) that

(35) $$[(U^2 - I)x, x] = 0 \quad \text{for every } x.$$

By Theorem 5 of [7] we conclude that $U^2 = I$.

One could give slightly different proofs of the "sufficiency" parts of Theorems 2 and 5 by using Theorem 6, showing that the given conditions characterize reflections.

From Theorem 6 above and Theorem 1 of [11] the next corollary is immediate.

COROLLARY 5. *Every adjoint abelian operator which is a multiple of an isometry is necessarily a scalar operator.*

In particular, as mentioned in the introduction, every adjoint abelian operator on $C(K)$ or L^p is scalar.

In conclusion, we raise the following question: On what Banach spaces is every adjoint abelian operator a real multiple of an isometry?

ACKNOWLEDGMENT. The authors would like to express their appreciation to Professor Earl Berkson for a valuable conversation in regard to Hermitian operators on $C(K)$.

BIBLIOGRAPHY

1. T. W. Ansah, *Hermitian operators of meromorphic type on Banach spaces,* Dissertation, University of Toronto, 1972.

2. C. Byrne, Jr. and F. E. Sullivan, *Contractive projections with contractive complement in L_p space,* J. Multivariate Anal. 2 (1972), 1–13. MR 46 #2412.

3. N. Dunford and J. T. Schwartz, *Linear operators.* I: *General theory,* Pure and Appl. Math., vol. 7, Interscience, New York, 1958. MR 22 #8302.

4. R. J. Fleming and J. E. Jamison, *Hermitian and adjoint abelian operators on certain Banach spaces,* Pacific J. Math. 52 (1974), 67–84.

5. D. Koehler and P. Rosenthal, *On isometries of normed linear spaces,* Studia Math. 36 (1970), 213–216. MR 43 #966.

6. J. Lamperti, *On the isometries of certain function-spaces,* Pacific J. Math. 2 (1958), 459–466. MR 21 #3764.

7. G. Lumer, *Semi-inner-product spaces,* Trans. Amer. Math. Soc. 100 (1961), 26–43. MR 24 #A2860.

8. ———, *Isometries of reflexive Orlicz spaces,* Ann. Inst. Fourier (Grenoble) **13** (1963), fasc. 1, 99–109. MR 28 #1485; erratum, 30, p. 1205.

9. H. L. Royden, *Real analysis,* Macmillan, London, 1968.

10. A. M. Sinclair, *Jordan homomorphisms and derivations on semisimple Banach algebras,* Proc. Amer. Math. Soc. 24 (1970), 209–214. MR 40 #3310.

11. J. G. Stampfli, *Roots of scalar operators,* Proc. Amer. Math. Soc. 13 (1962), 796–798. MR 25 #4362.

12. ———, *Adjoint abelian operators on Banach spaces,* Canad. J. Math. **21** (1969), 505–512. MR 39 #807.

13. K. W. Tam, *Isometries of certain function spaces,* Pacific J. Math. **31** (1969), 233–246.

14. E. Torrance, *Adjoints of operators on Banach spaces,* Dissertation, University of Illinois, 1968.

DEPARTMENT OF MATHEMATICS, MEMPHIS STATE UNIVERSITY, MEMPHIS, TENNESSEE 38152